

# Heat capacity of a self-gravitating spherical shell of radiations

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We study the heat capacity of a static system of self-gravitating radiations analytically in the context of general relativity. To avoid the complexity due to the conical singularity at the center, we excise the central part and replace it with a regular spherically symmetric distribution of matters in which specifications we are not interested. We assume that the mass inside the inner boundary and the locations of the inner and the outer boundaries are given. Then, we derive a formula relating the variations of physical parameters at the outer boundary with those at the inner boundary. Because there is only one free variation at the inner boundary, the variations at the outer boundary are dependent on each other, which determines the heat capacity. To get an analytic form for the heat capacity, we use the thermodynamic identity  $\delta S_{\text{rad}} = \beta \delta M_{\text{rad}}$ , which is derived from the variational relation of the entropy formula. Even if the radius of the inner boundary of the shell goes to zero, the heat capacity does not go to the form of the regular sphere.

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## I. INTRODUCTION

In 1980, Landau and Lifshitz [1] pointed out that system bound by long range forces might exhibit negative heat capacity even though the specific heat of each volume element is positive. Since then, many such examples were found, e.g., mainly stars and blackholes. A self-gravitating isothermal sphere also belongs to this class, which can be regarded as a model of a small dense nucleus of stellar systems [2]. Based on general relativity, the model was dealt by Sorkin, Wald, and Jiu [3] in 1981 as a spherically symmetric solution which maximizes entropy. Schmidt and Homann [4] called the geometry a ‘photon star’. The heat capacity and stability of the solution were further analyzed in Refs. [5–8]. Thereafter, the system has drawn attentions repeatedly in relation to the entropy bound [9, 10], blackhole thermodynamics [11], maximum entropy principle [12–14], holographic principle [15–17], and conjecture excluding blackhole firewalls [18]. A system of self-gravitating radiations in an Anti-de Sitter spacetime was also pursued [19–21]. Studies on the systems of self-gravitating perfect fluids are undergoing [22]. An interesting extension of the self-gravitating system was presented in Ref. [4, 23] where a conical singularity was included at the center as an independent mass source from the radiation. Some of the singular solutions were argued to have an interesting geometry, which is similar to an event horizon in the sense that  $1 - 2m(r)/r$  has a minimum value close to zero. Analytic approximation was tried to understand the situation that a blackhole is in equilibrium with the radiations [24]. It was also argued that the thermodynamics of a black hole in equilibrium implies the breakdown of Einstein equations on a macroscopic near-horizon shell [25]. The geometrical details of solutions having conical singularity were dealt in Ref. [26].

Let us consider a static spherically symmetric system of self-gravitating radiations confined in a spherical shell bounded by two boundaries located at  $r = r_-$  and  $r = r_+$  in the context of general relativity. For a generic time symmetric data, the initial value constraint equations become simply  ${}^{(3)}R = 16\pi\rho$ . As described in Ref. [3], this determines the spatial metric to be the form  $h_{ij}dx^i dx^j = [1 - 2m(r)/r]^{-1}dr^2 + r^2 d\Omega^2$ , where

$$m(r) = M_- + 4\pi \int_{r_-}^r \rho(r') r'^2 dr'. \quad (1)$$

Here  $M_-$  represents the mass inside the inner boundary at  $r_-$ . At present, we do not assume anything for the nature of  $M_-$  except for the spherical symmetry. Therefore, it can take negative value. The mass of the radiations in the shell is

$$M_{\text{rad}} = M_+ - M_-; \quad M_+ \equiv \lim_{r \rightarrow \infty} m(r), \quad (2)$$

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where  $M_+$  denotes the total mass of the solution. We neglect the energy density of the confining shell and assume that no matters lie outside of the outer boundary. We assume that the radiation does not interact with matters inside. The energy density of the radiation at the outer surface of the shell with local temperature  $T_+$  is

$$\rho(r_+) = \sigma T_+^4, \quad (3)$$

where  $\sigma$  is the Stefan-Boltzmann constant. The local heat capacity (LHC) for fixed volume with respect to the local temperature is

$$C_{\text{local}} \equiv \left( \frac{\partial M_{\text{rad}}}{\partial T_+} \right)_{r_{\pm}, M_-}. \quad (4)$$

In fact, the condition of fixed volume is not transparent because the metric  $g_{rr}$  contributes to the volume. In this work, we simply use the terminology to represent that the areal radius of the inner and outer boundaries do not change. Direct analytic calculation of the heat capacity is impossible because it requires to solve the corresponding equation of motions analytically, which was solved only numerically in the previous literatures. However, one can find a detour through the variation of entropy.

Formally, the entropy of the radiation with equation of state,  $\rho(r) = 3p(r)$ , can be obtained by integrating its entropy density over the volume, [3]

$$S_{\text{rad}} = \int_{r_-}^{r_+} L(r) dr; \quad L \equiv \frac{4(4\pi\sigma)^{1/4}}{3} \frac{r^{1/2} [m'(r)]^{3/4}}{\chi(r)}, \quad (5)$$

where  $\chi(r) \equiv \sqrt{1 - 2m(r)/r}$ . The variation of  $S_{\text{rad}}$  with respect to a small change of  $m(r)$  gives

$$\begin{aligned} \delta S_{\text{rad}} &= \int_{r_-}^{r_+} \left[ \frac{\partial L}{\partial m} - \frac{d}{dr} \frac{\partial L}{\partial m'} \right] \delta m dr + \left[ \frac{\partial L}{\partial m'} \delta m \right]_{r_-}^{r_+} \\ &= \int_{r_-}^{r_+} \frac{\delta S_{\text{rad}}}{\delta m} \delta m dr + \beta_+ \delta M_+ - \beta_- \delta M_-, \end{aligned} \quad (6)$$

where

$$\beta_{\pm} \equiv \left( \frac{\partial L}{\partial m'} \right)_{r \rightarrow r_{\pm}} = \left[ \frac{r^{1/2}}{\chi} \left( \frac{4\pi\sigma}{m'(r)} \right)^{1/4} \right]_{r \rightarrow r_{\pm}}. \quad (7)$$

Noting the relation of mass with the surface energy density in Eqs. (1) and (3), the local temperature  $T_+$  is related with  $\beta \equiv \beta_+$  as

$$\beta^{-1} = \chi_+ T_+. \quad (8)$$

One may introduce the metric component  $g_{tt}$  so that the local temperature at  $r$  is given by the Doppler-shifted temperature as,

$$\sqrt{-g_{tt}(r)} T(r) = \sqrt{-g_{tt}(r)} \left( \frac{\rho(r)}{\sigma} \right)^{1/4} = \beta^{-1}. \quad (9)$$

This result can also be obtained by solving the Einstein's equation directly. This equation indicates that  $\beta^{-1}$  is the global temperature measured at the asymptotic region. On the other hand,  $\beta_-$  is not directly related with the local temperature  $T_-$  by the relation in Eq. (9) but is related by

$$\beta_-^{-1} = \sqrt{\frac{r_+}{r_-}} \chi_- T_-. \quad (10)$$

Given the temperature  $\beta^{-1}$ , the heat capacity for fixed volume of the shell is defined by

$$C_V = \left( \frac{\partial M_{\text{rad}}}{\partial \beta^{-1}} \right)_{r_+} = \left( \frac{\partial M_+}{\partial \beta^{-1}} \right)_{r_+} - \left( \frac{\partial M_-}{\partial \beta^{-1}} \right)_{r_+}. \quad (11)$$

At the present case, the second term vanishes because  $M_-$  is held.

Introducing scale invariant variables  $u$  and  $v$  as

$$u \equiv \frac{2m(r)}{r}, \quad v \equiv \frac{dm(r)}{dr} = 4\pi r^2 \rho(r) = 4\pi \sigma r^2 T(r)^4, \quad (12)$$

the variational equation  $\delta S_{\text{rad}}/\delta m = 0$  becomes a first order differential equation for  $u$  and  $v$ ,

$$\frac{dv}{du} = f(u, v) \equiv \frac{2v(1 - 2u - 2v/3)}{(1 - u)(2v - u)}. \quad (13)$$

This equation is equivalent to the general relativistic Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium for a radiation. The allowed range of  $(u, v)$  is  $u < 1$  and  $v \geq 0$ , where each inequality represents the fact that the spacetimes is static and the energy density of radiation is non-negative, respectively. Integrating this equation on the  $(u, v)$  plane, solution curves were found in Refs. [3, 24]. Any solution curve will be parallel to the  $u$ -axis when it crosses the line

$$P : 2u + \frac{2v}{3} = 1, \quad (14)$$

and is parallel to the  $v$ -axis when it crosses the line

$$H : u = 2v. \quad (15)$$

The solution curve eventually converges to the point  $\mathcal{R} \equiv (3/7, 3/14)$  where the two lines  $P$  and  $H$  meet. A specific solution curve  $C_\nu$  is characterized by

$$\nu \equiv 1 - u_H = 1 - \frac{2m(r_H)}{r_H}, \quad (16)$$

the orthogonal distance of  $C_\nu$  from the line  $u = 1$  on the  $(u, v)$  plane [26]. Here the subscript  $H$  represents the point where  $C_\nu$  crosses  $H$ , which is the position of the approximate horizon defined by a surface that resembles an apparent horizon [26]. We quote the name ‘approximate horizon’ from Ref. [25]. The value of  $\nu$  varies from zero, the solution corresponding to the formation of an event horizon, to  $\nu_r \approx 0.50735$ , the everywhere regular solution. A given solution curve is parameterized by a scale invariant variable

$$\xi \equiv \log \frac{r}{r_H}. \quad (17)$$

Therefore, the physically relevant region of  $(u, v)$  plane can be equivalently coordinated by using the set  $(\nu, \xi)$ . Now, a specific sphere solution of radiations can be characterized by choosing a boundary point on a curve  $C_\nu$  after picking the radius of the boundary  $r_+$ .

A given spherical shell of radiation can be denoted by four different numbers,  $(\nu, r_H, e^{\xi_+}, e^{\xi_-})$ , representing a specific solution curve, the radius of the approximate horizon for the solution curve, and the positions of the inner and the outer boundaries relative to the approximate horizon, respectively. The scale invariant variables at the outer boundary are related with the total mass, local temperature, and the radius as

$$r_+ = r_H e^{\xi_+}, \quad u_+ \equiv u_\nu(\xi_+) = \frac{2M_+}{r_+}, \quad v_+ \equiv v_\nu(\xi_+) = 4\pi r_+^2 \rho(r_+) = 4\pi \sigma r_+^2 T_+^4, \quad (18)$$

where we put the subscript  $\nu$  to  $u$  and  $v$  to represent the specific solution curve  $C_\nu$ . The scale invariant values at the inner boundary are given by

$$r_- = r_H e^{\xi_-}, \quad u_- \equiv u_\nu(\xi_-) = \frac{2M_-}{r_-}, \quad v_- \equiv v_\nu(\xi_-) = 4\pi r_-^2 \rho(r_-). \quad (19)$$

In this work, the value of  $u_-$  and  $r_-$  are held. On the other hand, the value of  $v_-$  should be determined by tracing in the solution curve  $C_\nu$  from the data at the outer boundary.

Even though the static solution of the self-gravitating radiations were studied well, its stability needs additional study. To achieve this purpose we study its heat capacity. In Sec. II, we first derive the relation between the variations of  $(u, v)$  and those of  $(\nu, \xi)$ . By using the fact that  $\delta\nu$  and  $\delta\xi$  are the same if  $(u_+, v_+)$  and  $(u_-, v_-)$  are on one solution curve  $C_\nu$ , we relate the variations of physical parameters at the outer boundary with those at the inner boundary. In Sec. III, we calculate the (local) heat capacity for fixed volume from the variational equation of entropy. We show that the general heat capacity is located in the middle of the two extreme forms, that of the regular solution and that of other extreme. In Sec. IV, we study various limiting behaviors of the heat capacity. We summarize the results in Sec. V.

## II. VARIATIONS OF THE SCALE INVARIANT VARIABLES

The difficulty in calculating the heat capacity of spherical shell of matters lays on the fact that the physical parameters at the inner boundary are dependent on those at the outer boundary, which exact relation needs analytic solutions. Rather than searching for an exact analytic solution, we find a variational relation between them. Because  $\delta r_- = 0 = \delta M_-$ , we have  $\delta u_- = 0$  leaving only  $\delta v_-$  be dependent on the variations at the outer boundary. We study how to relate the variations at the outer boundary with those at the inner boundary in a general setting. To do this, we find the variations  $\delta \nu$  and  $\delta \xi$  corresponding to the variations  $(\delta u_+, \delta v_+)$ . Then, we use (i) The variation  $\delta \nu$  is independent of the position of  $(u_-, v_-)$  if it is on the same solution curve  $C_\nu$  with  $(u_+, v_+)$ . (ii) The variation  $\delta \xi = \delta r/r - \delta r_H/r_H$  is also independent of the position of  $(u_-, v_-)$  if  $r = r_\pm$  are held. (iii)  $\nu$  and  $\xi$  defines an orthogonal coordinates system which is equivalent to  $(u, v)$  physically.

### A. Generic variations

Let us consider the variation  $\delta \xi$  and  $\delta \nu$ , which represent the variations parallel to and orthogonal to the solution curve  $C_\nu$ , respectively. The two variations are orthogonal to each other and define most general changes of boundary points,  $(u, v) \equiv (u_\pm, v_\pm)$ , on  $C_\nu$ . An important point here is that  $\nu$  is independent of the choice of the boundary point on  $C_\nu$  by definition and the variation of  $\xi_\pm = \log(r_\pm/r_H)$  is required to be dependent only on the change of  $r_H$  because  $r_\pm$  are held. Therefore, we have  $\delta \xi_+ = \delta r_H/r_H = \delta \xi_-$ . In this subsection, we omit the subscript  $\pm$  for notational simplicity.

From Eqs. (12) and (17), we get  $(\partial u/\partial \xi) = 2v - u$ . Therefore, by using Eq. (13), the tangent along the solution curve is given by

$$\frac{\partial}{\partial \xi} = (2v - u) \frac{\partial}{\partial u} + (2v - u) f(u, v) \frac{\partial}{\partial v}. \quad (20)$$

On the other hand, the derivative orthogonal to the solution curve  $C_\nu$  can be written as

$$\frac{\partial}{\partial \nu} = -B(u, v) f(u, v) \frac{\partial}{\partial u} + B(u, v) \frac{\partial}{\partial v}, \quad (21)$$

where we use  $\partial/\partial \xi \perp \partial/\partial \nu$  and  $B$  is a local function of  $(u, v)$  defined by

$$B(u, v) \equiv \left( \frac{\partial v}{\partial \nu} \right)_\xi. \quad (22)$$

The explicit functional form of  $B$  will be determined later in this section in Eq. (35) from consistency. From Eqs. (20) and (21), we determine  $\delta u$  and  $\delta v$  in terms of  $\delta \nu$  and  $\delta \xi$  as,

$$\delta u = (2v - u) \delta \xi - B f \delta \nu, \quad \delta v = (2v - u) f \delta \xi + B \delta \nu. \quad (23)$$

Inverting Eq. (23), the variation  $\delta \nu$  and  $\delta \xi$  are given by

$$\delta \nu = \frac{-f \delta u + \delta v}{B(1 + f^2)}, \quad \delta \xi = \frac{\delta u + f \delta v}{(2v - u)(1 + f^2)}. \quad (24)$$

Let us see the results at the point  $r = r_H$  where  $u_H = 2v_H$ . At that point,  $\delta \xi$  and  $\delta \nu$  are parallel to  $\delta v$  and  $\delta u$ , respectively. Therefore,  $(\delta u/\delta \xi)_{r=r_H} = 0$ ,  $(\delta v/\delta \nu)_{r=r_H} = 0$ . In addition, from Eq. (23),

$$\delta v_H = [(2v - u) f(u, v)]_{r \rightarrow r_H} \delta \xi = \frac{14v_H/3 - 1}{1 - 2v_H} v_H \delta \xi, \quad \delta u_H = - \left[ \lim_{r \rightarrow r_H} B(u, v) f(u, v) \right] \delta \nu. \quad (25)$$

From the first equation, one notes that  $\delta v_H$  diverges as  $v_H \rightarrow 1/2$ , which corresponds to the limit of forming an event horizon. The second equation, by using Eq. (16), determines the normalization of  $B$  to be

$$\lim_{r \rightarrow r_H} f(u, v) B(u, v) = 1. \quad (26)$$

Later in this section, we finalize the function  $B$  from this normalization condition. Because  $f(u, v)$  diverges on  $H$ , the value of  $B$  will vanish there.

## B. Variations at the inner and the outer boundaries

By using the fact that  $\delta\nu$  and  $\delta\xi$  are independent of the position on a given solution curve  $C_\nu$ , we relate the variations at the outer boundary with those at the inner boundary. From Eqs. (23) and (24), the variation of  $u_-$  can be written by the variations at the outer boundary as

$$\begin{aligned}\delta u_- &= -B_- f_- \delta\nu + (2v_- - u_-) \delta\xi \\ &= \frac{f_+ f_-}{1 + f_+^2} \left( \frac{B_-}{B_+} + \frac{1}{f_+ f_-} \frac{2v_- - u_-}{2v_+ - u_+} \right) \delta u_+ + \frac{f_-}{1 + f_+^2} \left( -\frac{B_-}{B_+} + \frac{f_+}{f_-} \frac{2v_- - u_-}{2v_+ - u_+} \right) \delta v_+, \end{aligned} \quad (27)$$

where  $B_\pm$  and  $f_\pm$  stand for  $B(u_\pm, v_\pm)$  and  $f(u_\pm, v_\pm)$ , respectively. Using the definition of the LHC in Eq. (4) after dividing Eq. (27) by  $\delta M_+$ , we get

$$\left( f_+ - \frac{2r_+ v_+}{T_+} C_{\text{local}}^{-1} \right) \frac{B_-}{B_+} = - \left( \frac{1}{f_+} + \frac{2r_+ v_+}{T_+} C_{\text{local}}^{-1} \right) \frac{f_+}{f_-} \frac{2v_- - u_-}{2v_+ - u_+}, \quad (28)$$

where we use  $(\partial u_- / \partial M_+)_{r_\pm, M_-} = 0$  because  $r_-$  and  $M_-$  are held. In a similar manner, the variation of  $v_-$  can be written by means of the variations at the outer boundary as

$$\begin{aligned}\delta v_- &= B_- \delta\nu + (2v_- - u_-) f_- \delta\xi \\ &= \frac{f_+}{1 + f_+^2} \left( -\frac{B_-}{B_+} + \frac{(2v_- - u_-) f_-}{(2v_+ - u_+) f_+} \right) \delta u_+ + \frac{1}{1 + f_+^2} \left( \frac{B_-}{B_+} + \frac{(2v_- - u_-) f_- f_+}{2v_+ - u_+} \right) \delta v_+. \end{aligned} \quad (29)$$

Using Eq. (4) after dividing Eq. (29) by  $\delta M_+$ , we get

$$\left( \frac{\partial v_-}{\partial M_+} \right)_{r_\pm, M_-} = \frac{2}{r_+} \frac{1}{1 + f_+^2} \left[ - \left( f_+ - \frac{2r_+ v_+}{T_+} C_{\text{local}}^{-1} \right) \frac{B_-}{B_+} + f_+ f_- \frac{2v_- - u_-}{2v_+ - u_+} \left( \frac{1}{f_+} + \frac{2r_+ v_+}{T_+} C_{\text{local}}^{-1} \right) \right]. \quad (30)$$

Putting Eq. (28) to Eq. (30), one gets

$$\left( \frac{\partial v_-}{\partial M_+} \right)_{r_\pm, M_-} = \frac{2}{r_+} \frac{f_- + f_-^{-1} \frac{2v_- - u_-}{2v_+ - u_+}}{f_+ + f_+^{-1} \frac{2v_- - u_-}{2v_+ - u_+}} \left[ \frac{1}{f_+} + \frac{2r_+ v_+}{T_+} C_{\text{local}}^{-1} \right]. \quad (31)$$

Once we get the LHC we can obtain the function  $B$  from Eq. (28) in addition to the relation between the outer boundary and the inner boundary through Eq. (31).

Even though  $C_{\text{local}}$  will be identified later in Eq. (44), we use the result here to determine the function  $B$ . Equation (28) gives

$$\sqrt{\frac{r_-}{r_+}} \frac{f_- B_-}{f_+ B_+} = \frac{(2v_- - u_-) A_-}{(2v_+ - u_+) A_+}, \quad (32)$$

where  $A_\pm \equiv A(u_\pm, v_\pm)$  and

$$A(u, v) \equiv \frac{v^{3/4}}{\chi} \frac{f}{(2v - u)^2 (1 + f^2)} = \frac{v^{3/4} \chi}{2v - u} \frac{F}{F^2 + G^2}. \quad (33)$$

In the second equality, we use  $f = F/G$  with

$$F \equiv 2v \left( 1 - 2u - \frac{2v}{3} \right), \quad G \equiv (1 - u)(2v - u). \quad (34)$$

Note that the function  $(2v - u)A(u, v)$  is regular on the whole range of physical interest except for the point  $\mathcal{R}$ , which corresponds to the asymptotic infinity  $r \rightarrow \infty$  of all solution curves.  $B(u, v)$  must be a local function of  $(u, v)$ . Therefore, Eq. (32) determines  $B(u, v)$  up to a proportionality constant which is a function of  $\nu$  only,

$$B(u, v) = \alpha_\nu \sqrt{\frac{r_H}{r}} \frac{(2v - u)A(u, v)}{f(u, v)} = \alpha_\nu \sqrt{\frac{r_H}{r}} \frac{v^{3/4} \chi G}{F^2 + G^2}. \quad (35)$$

Here  $\nu$  and  $r/r_H = e^\xi$  are implicitly dependent on  $u$  and  $v$ . It goes to zero as expected in Eq. (26).  $B(u, v)$  diverges on  $\mathcal{R}$ . The proportionality constant  $\alpha_\nu$  can be fixed by using Eq. (26), after choosing  $(u_+, v_+) = (u_H, v_H)$  and  $(u_-, v_-) = (u, v)$ , to be

$$\alpha_\nu = \lim_{r \rightarrow r_H} \frac{1}{(2v - u)A} = \frac{2^{3/4} (7\nu - 4)(1 - \nu)^{1/4}}{3 \nu^{1/2}}. \quad (36)$$

Note that  $\alpha_\nu$  is negative definite because  $\nu$  is restricted to be  $0 < \nu \leq \nu_r < 4/7$ .

### III. CALCULATION OF THE HEAT CAPACITY

Once the equation of motion in Eq. (13) holds, the variational equation (6) of the entropy presents the first law of thermodynamics as

$$\delta S_{\text{rad}} = \beta \delta M_+ - \beta_- \delta M_- = \beta \delta M_{\text{rad}} + (\beta - \beta_-) \delta M_- . \quad (37)$$

When  $\delta M_- = 0$ , one can identify  $\beta \equiv \beta_+$  as the inverse temperature measured at the asymptotic region. In a general situation,  $\delta M_-$  could be dependent on the choice of the values at the outside and the radius of the inner boundary as

$$\delta M_- = \frac{\partial M_-}{\partial M_+} \delta M_+ + \frac{\partial M_-}{\partial \beta} \delta \beta + \frac{\partial M_-}{\partial r_+} \delta r_+ + \frac{\partial M_-}{\partial r_-} \delta r_- . \quad (38)$$

To obtain heat capacities other than that conserve the mass inside the inner boundary,  $\delta M_-$  should be dealt in this way.

Fortunately, the integration in Eq. (5) can be executed to give an analytic form for the entropy of the radiation [3, 6],

$$S_{\text{rad}} \equiv S_+ - S_-, \quad S_{\pm}(u_{\pm}, v_{\pm}, r_{\pm}) = \frac{r_{\pm}^{3/2}}{3\chi_{\pm}} \left( \frac{4\pi\sigma}{v_{\pm}} \right)^{1/4} \left( \frac{2v_{\pm}}{3} + u_{\pm} \right) = \frac{r_{\pm}\beta_{\pm}}{3} \left( \frac{2v_{\pm}}{3} + u_{\pm} \right) . \quad (39)$$

Note that  $S_{\pm}$  does not represent the entropy of the objects inside  $r_{\pm}$ . For later convenience, we put the derivative of  $S_{\pm}$  as

$$\beta_{\pm}^{-1} dS_{\pm} = \frac{1}{2} \left( \frac{2v_{\pm}}{3} + u_{\pm} \right) dr_{\pm} + \frac{r_{\pm}}{6} \frac{2 - u_{\pm} + 2v_{\pm}/3}{1 - u_{\pm}} du_{\pm} + \frac{r_{\pm}}{12} \frac{2v_{\pm} - u_{\pm}}{v_{\pm}} dv_{\pm} . \quad (40)$$

If we consider on-shell change [ $du$  and  $dv$  are related by Eq. (13)], we get the first law of thermodynamics,  $dM = \beta^{-1} dS - p(4\pi r^2) dr$ .

#### A. Local heat capacity

Now, let us calculate the LHC for fixed volume in Eq. (4). Direct calculation of the heat capacity needs to solve the equation of motion (13) from  $r_-$  to  $r_+$ , which is impossible analytically. On the other hand, the entropy has an exact analytic expression. Therefore, it would be better to use Eq. (37) to obtain the heat capacity. When  $M_-$  is held, Eq. (37) becomes

$$0 = \left( \frac{\partial S_{\text{rad}}}{\partial M_{\text{rad}}} \right)_{r_{\pm}, M_-} - \beta = \left( \frac{\partial S_+}{\partial M_+} \right)_{r_+} - \beta - \left( \frac{\partial S_-}{\partial M_+} \right)_{r_{\pm}, M_-} , \quad (41)$$

where  $S_{\text{rad}}$  is given in Eq. (39). Because  $r_{\pm}$  and  $M_-$  are held,  $S_+$  and  $S_-$  are local functions of  $(u_+, v_+)$  and  $v_-$ , respectively.

Before dealing complex general cases, let us review how the heat capacity for a self-gravitating radiation sphere with regular center was calculated in Ref. [5] by choosing  $\nu = \nu_r$  and  $r_- = 0$ . Because  $S_- = 0$ , the last term in the right-hand side of Eq. (41) vanishes. Noting  $r_+$  is held, by using Eqs. (18) and (40), the right-hand side of Eq. (41) becomes

$$\begin{aligned} \left( \frac{\partial S_+}{\partial M_+} \right)_{r_+} - \beta &= \frac{2}{r_+} \left( \frac{\partial S_+}{\partial u_+} \right)_{r_+, v_+} + \frac{4v_+}{T_+} \left( \frac{\partial T_+}{\partial M_+} \right)_{r_+} \left( \frac{\partial S_+}{\partial v_+} \right)_{r_+, u_+} - \beta \\ &= \frac{\beta r_+ (2v_+ - u_+)}{12v_+} \left[ -\frac{2f_+}{r_+} + \frac{4v_+}{T_+} \left( \frac{\partial T_+}{\partial M_+} \right)_{r_+} \right] . \end{aligned} \quad (42)$$

For a regular solution, there remains only one free degree of freedom in the physical parameters at the outer boundary because the size  $r_+$  is held. Therefore, the variations  $\delta u_+$  and  $\delta v_+$  must be dependent on each other, which relation determines the LHC. Now, the LHC for the regular solution is given after setting Eq. (42) to zero:

$$C_{\text{local}}^{\text{reg}} = \left( \frac{\partial M_+}{\partial T_+} \right)_{r_+} = \frac{2r_+ v_+}{T_+ f_+} . \quad (43)$$

The LHC for self-gravitating regular sphere of radiations diverges when the solution curve intersects  $P$  and vanishes when the solution curve intersects  $H$ .

To obtain the LHC for a general case with  $r_- \neq 0$ , the effect of  $S_-$  should also be taken into account. Equating Eq. (41) by using Eqs. (31), (39), (40), (42) and using  $\left(\frac{\partial S_-}{\partial M_+}\right)_{r_{\pm}, M_-} = \left(\frac{\partial v_-}{\partial M_+}\right)_{r_{\pm}} \left(\frac{\partial S_-}{\partial v_-}\right)_{r_-, u_-}$ , we get

$$C_{\text{local}} = \frac{2r_+v_+}{T_+f_+} \frac{1 - \mathfrak{A}}{1 + f_+^{-2}\mathfrak{A}}, \quad \mathfrak{A} \equiv \sqrt{\frac{r_-}{r_+}} \frac{A_+}{A_-}, \quad (44)$$

where we use Eqs. (8), (9), (10), and (33) to simplify the form of the equation. In the limit  $r_- \rightarrow r_+$ , the LHC goes

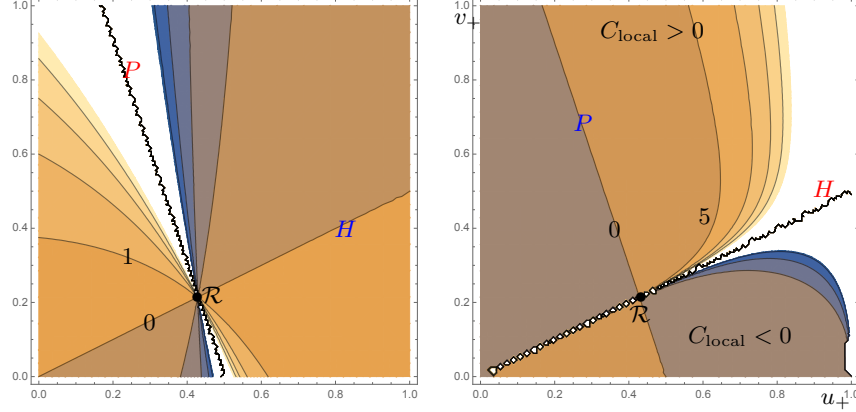


FIG. 1:  $T_+C_{\text{local}}/r_+$  for the limit of  $\mathfrak{A} \rightarrow 0$  (L) and  $\mathfrak{A} \rightarrow \infty$  (R). On each figure, singularities of the LHC are plotted as wiggly line and marked by a red character.

to zero as expected. The LHC is formally the same as that of the regular sphere in Eq. (43) when  $\mathfrak{A} \rightarrow 0$ , which limit is shown in the left panel of Fig. 1. It is singular on  $P$  and vanishes on  $H$ . When viewed from the clockwise direction centered on  $\mathcal{R}$ , the LHC takes positive values from  $H$  to  $P$  and negative values elsewhere. This limit happens when  $A_- \rightarrow \infty$  which is achieved if the inner boundary is located on the approximate horizon, i.e.,  $r_- = r_H$ . To outward seeming, the limit  $r_- \rightarrow 0$  also leads to the result. However, as will be shown in subSec. IV B, this is not the case except for the regular solution because  $A_-$  behaves nontrivially. The opposite limit  $\mathfrak{A} \rightarrow \infty$  is achieved when  $A_- \rightarrow 0$ . This limit happens if the inner boundary is located on the line  $P$ . This case is displayed in the right panel of Fig. 1. In this case, the LHC becomes

$$C_{\text{local}}^P \approx -\frac{2r_+v_+f_+}{T_+}. \quad (45)$$

This is singular on the line  $H$  and vanishes on the line  $P$ . When viewed from the clockwise direction, it takes positive values from  $P$  to  $H$  and negative values elsewhere.

Writing the LHC (44) explicitly in terms of  $(u_+, v_+)$ ,

$$C_{\text{local}} = \frac{2r_+v_+}{T_+} \frac{G_+(F_+^2 + G_+^2) - \frac{v_+^{3/4}\chi^3}{A_-(r_+/r_-)^{1/2}}F_+}{F_+(F_+^2 + G_+^2) + \frac{v_+^{3/4}\chi^3}{A_-(r_+/r_-)^{1/2}}G_+}. \quad (46)$$

Note that the denominator vanishes on the curve

$$S: 2\left(1 - 2u_+ - \frac{2v_+}{3}\right)(F_+^2 + G_+^2) + \frac{v_+^{-1/4}\chi^5}{A_-(r_+/r_-)^{1/2}}(2v_+ - u_+) = 0. \quad (47)$$

On this curve, the LHC is singular. The curve  $S$  passes the point  $\mathcal{R}$  along the line  $H$  because  $F_+ \rightarrow 0$  and  $G_+ \rightarrow 0$  at  $\mathcal{R}$  leaving the last term in Eq. (47) as the first nontrivial corrections. Equation (47) indicates that  $S$  must be around  $P$  for  $A_-(r_+/r_-)^{1/2} \gg 1$  and around  $H$  for  $A_-(r_+/r_-)^{1/2} \ll 1$ . These behaviors are manifest in the left and right panel of Fig. 2, respectively. As  $F_+$  or  $G_+$  are larger, i.e.  $v_+$  increases, the singular curve gradually approaches the

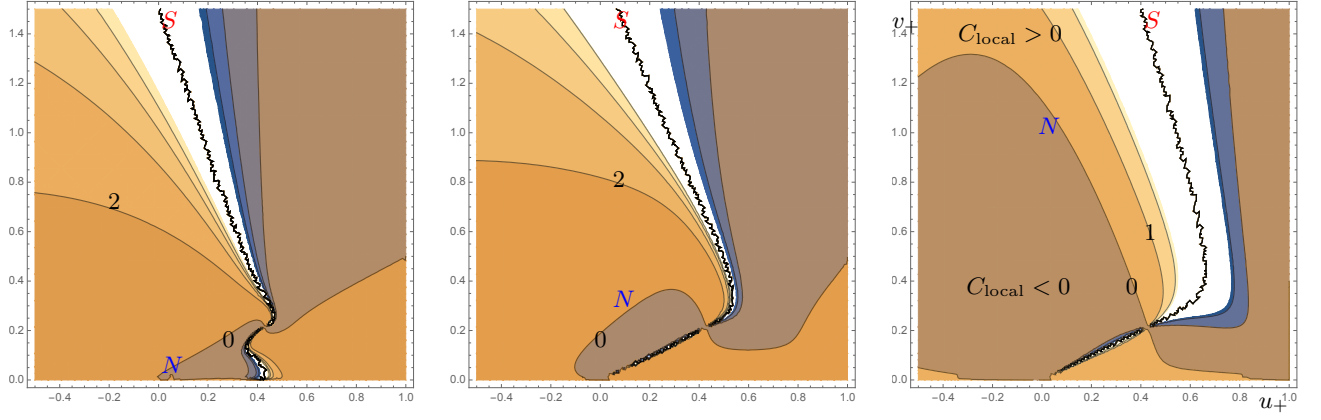


FIG. 2:  $T_+ C_{\text{local}} / (r_+)$  with respect to the location of the outer boundary. The heat capacity for  $A_- r_+^{1/2} = 20, 1$ , and  $1/20$ , respectively from the left.

line  $P$ . Combining the pictures in Fig. 1 with those in Fig. 2, one may find that the singular curve  $S$  gradually change from the line  $P$  to the line  $H$  as  $A_-(r_+/r_-)^{1/2}$  decreases. The numerator of Eq. (46) vanishes on the curve

$$N : (2v_+ - u_+)(F_+^2 + G_+^2) - \frac{2\sqrt{1 - u_+ v_+}^{7/4}}{A_-(r_+/r_-)^{1/2}} \left(1 - 2u_+ - \frac{2v_+}{3}\right) = 0. \quad (48)$$

The LHC changes signature on the null curve  $N$ . The null curve passes the point  $\mathcal{R}$  along the line  $P$  because  $F_+, G_+ \rightarrow 0$  at  $\mathcal{R}$  leaving the last term in Eq. (48) as the first nontrivial corrections. Equation (48) indicates that the curve  $N$  must overlaps with  $H$  for  $A_-(r_+/r_-)^{1/2} \gg 1$  and with  $P$  for  $A_-(r_+/r_-)^{1/2} \ll 1$ , respectively. Combining the pictures in Fig. 1 with those in Fig. 2, one may notice the the null curve gradually changes from the line  $H$  to the line  $P$  as  $A_-(r_+/r_-)^{1/2}$  decreases. When viewed from the clockwise direction centered at  $\mathcal{R}$ , the LHC takes positive values from  $N$  to  $S$  and negative values elsewhere.

### B. Heat capacity

The heat capacity (11) with respect to the global temperature  $\beta^{-1}$  can be obtained based on the value of LHC. At the present case, the second term in Eq. (11) vanishes because  $M_-$  is held. Let us calculate the first term in the right hand side. Varying Eq. (8), we get

$$\delta\beta^{-1} = \frac{M_+ T_+}{\chi^2 r_+^2} \delta r_+ - \frac{T_+}{r_+ \chi} \delta M_+ + \chi \delta T_+, \quad (49)$$

where we regard  $\beta$  as a function of  $T_+$ ,  $M_+$ , and  $r_+$ . Generally, the three variations  $\delta T_+$ ,  $\delta M_+$ , and  $\delta r_+$  are independent. However, if the state inside the inner boundary is invariant under the changes of the physical parameters at the outer boundary, i.e.  $\delta r_- = 0 = \delta M_-$ , only two of the three variations will be independent. If the size of the shell is held,  $\delta r_+ = 0$ , only one independent variation remains. In this case, the variations  $\delta T_+$  and  $\delta M_+$  must be related. The heat capacity is given by dividing both sides of Eq. (49) with  $\delta\beta^{-1}$ ,

$$C_V = \left( \frac{\partial M_{\text{rad}}}{\partial \beta^{-1}} \right)_{r_{\pm}, M_-} = \frac{r_+ \chi^2}{\beta^{-1}} \frac{1}{(r_+ \chi^2 / T_+) C_{\text{local}}^{-1} - 1}. \quad (50)$$

For the case of a regular sphere, inserting the result in Eq. (43) to Eq. (50), the heat capacity becomes

$$C_V^{\text{reg}} = \left( \frac{\partial M_+}{\partial \beta^{-1}} \right)_{r_+} = -\frac{r_+ \chi^2}{\beta^{-1}} \frac{2v_+ - u_+}{8v_+/3 - 1 + u_+}. \quad (51)$$

The heat capacity changes sign on the line  $H$  and is singular on the line

$$Q : \frac{8v_+}{3} + u_+ = 1. \quad (52)$$



In a general case, inserting Eq. (44) to Eq. (50), the heat capacity for the shell is given by

$$C_V = \frac{r_+ \chi^2}{\beta^{-1}} \frac{1 - \mathfrak{A}}{\chi^2 f_+ / (2v_+) - 1 + (\chi^2 / (2v_+ f_+) + 1) \mathfrak{A}}. \quad (53)$$

In the limit  $\mathfrak{A} \rightarrow 0$ , the heat capacity reproduces that of the regular sphere in Eq. (51). On the opposite limit  $\mathfrak{A} \rightarrow \infty$ ,

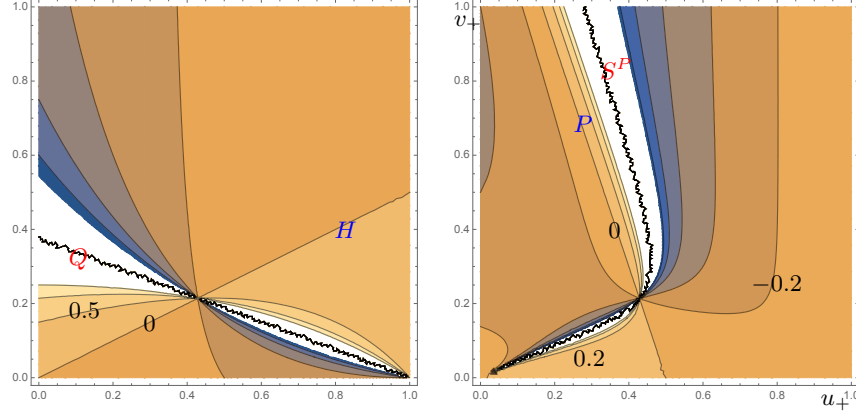


FIG. 3:  $T_+ C_{\text{local}} / r_+$  for the limit of  $\mathfrak{A} \rightarrow 0$  (L) and  $\mathfrak{A} \rightarrow \infty$  (R).

we have

$$C_V^P \equiv \lim_{\mathfrak{A} \rightarrow \infty} C_V = -\frac{r_+ \chi^2}{\beta^{-1}} \frac{4v_+^2(1 - 2u_+ - 2v_+/3)}{(1 - u_+)^2(2v_+ - u_+) + 4v_+^2(1 - 2u_+ - 2v_+/3)}. \quad (54)$$

As shown in the right panel of Fig. 3, the heat capacity in this limit vanishes on the line  $P$  and is singular on the curve

$$S^P : (1 - u_+)^2(2v_+ - u_+) + 4v_+^2(1 - 2u_+ - 2v_+/3) = 0. \quad (55)$$

The curve passes the point  $\mathcal{R}$ . It overlaps with the line  $H$  for  $v_+ \ll 1$  and approaches the line

$$1 - u_+ = \beta v_+, \quad \beta = \frac{2}{3} \left( -1 - \frac{2^{2/3}}{(4 + 3\sqrt{2})^{1/3}} + (8 + 6\sqrt{2})^{1/3} \right) \approx 0.5062 \quad (56)$$

for  $v_+ \gg 1$ . These behaviors are manifest in the right panel of Fig. 3.

Writing the heat capacity (53) explicitly in terms of  $(u_+, v_+)$ , we get

$$C_V = \frac{r_+ \chi^2}{\beta^{-1}} \frac{(2v_+ - u_+)(F_+^2 + G_+^2) - \frac{v_+^{3/4} \chi_+}{A_-(r_+/r_-)^{1/2}} F_+}{(1 - u_+ - \frac{8v_+}{3})(F_+^2 + G_+^2) + \frac{v_+^{3/4} \chi_+}{A_-(r_+/r_-)^{1/2}} [F_+ + \chi_+^2 G_+ / (2v_+)]}. \quad (57)$$

The denominator of Eq. (57) vanishes on the curve given by

$$S' : \left(1 - u_+ - \frac{8v_+}{3}\right)(F_+^2 + G_+^2) + \frac{v_+^{-1/4} \chi_+}{2A_-(r_+/r_-)^{1/2}} (\chi_+^2 G_+ + 2v_+ F_+) = 0. \quad (58)$$

On this curve, the heat capacity is singular. The curve passes  $\mathcal{R}$  along the curve  $S^P$  because  $F_+ \rightarrow 0$  and  $G_+ \rightarrow 0$  at  $\mathcal{R}$  leaving the last term in Eq. (58) as the first nontrivial corrections. Equation (58) indicates that the singular curve  $S'$  must be around the line  $Q$  and the curve  $S^P$  for  $A_-(r_+/r_-)^{1/2} \gg 1$  and  $A_-(r_+/r_-)^{1/2} \ll 1$ , respectively. These behaviors are manifest in the left and right panels of Fig. 4, respectively. As  $F_+$  or  $G_+$  are larger, i.e.,  $\chi$  or  $v_+$  increases, the singular curve gradually approaches the line  $Q$ . Combining the pictures in Figs. 3 and 4, one may find the singular curve gradually change from the line  $Q$  to the curve  $S^P$  as  $A_-(r_+/r_-)^{1/2}$  decreases.

The numerator of Eq. (57) vanishes on the curve

$$N' : (2v_+ - u_+)(F_+^2 + G_+^2) - \frac{v_+^{7/4} \chi}{A_-(r_+/r_-)^{1/2}} \left(1 - 2u_+ - \frac{2v_+}{3}\right) = 0. \quad (59)$$

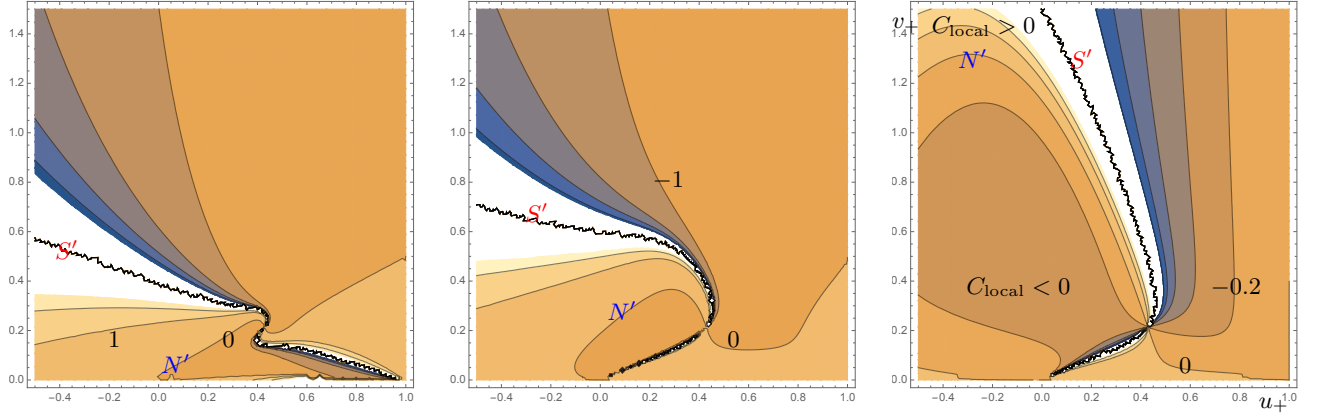


FIG. 4:  $\beta_+^{-1} C_V / (r_+)$  with respect to the location of the outer boundary. The heat capacity for  $A_-(r_+/r_-)^{1/2} = 20, 1$ , and  $1/20$ , respectively from the left.

The heat capacity changes sign on  $N'$ . The curve passes the point  $\mathcal{R}$  along the line  $P$  because  $F_+ \rightarrow 0$  and  $G_+ \rightarrow 0$  at  $\mathcal{R}$  leaving the last term in Eq. (59) as the first nontrivial corrections. Equation (59) indicates that the curve  $N'$  must be around the line  $H$  for  $A_-(r_+/r_-)^{1/2} \gg 1$  and around  $P$  for  $A_-(r_+/r_-)^{1/2} \ll 1$ , respectively. These behaviors are manifest in the left and right panels of Fig. 4, respectively. Combining the pictures in Fig. 3 and Fig. 4, one may find the the curve  $N'$  gradually changes from the line  $H$  to the line  $P$  as  $A_-(r_+/r_-)^{1/2}$  decreases. When viewed from the clockwise direction centered at  $\mathcal{R}$ , the heat capacity takes positive values from  $N'$  to  $S'$  and negative values elsewhere.

#### IV. VARIOUS LIMITS

Because the functional forms of the LHC and the heat capacity are complicated, we present various physically interesting limits to improve our understanding on the system.

##### A. Thin shell limit

We first consider the thin shell limit,  $\delta r \equiv r_+ - r_- \ll r_+ \approx r$ ,  $\delta u \equiv u_+ - u_- \ll u_+$ , and  $\delta v \equiv v_+ - v_- \ll v_+$ . By using

$$\delta r = r \delta \xi = \frac{r \delta u}{2v - u}, \quad \delta v = f(u, v) \delta u,$$

the heat capacity and the LHC in this limit take the form,

$$\begin{aligned} \frac{\chi^{-1} C_V}{\delta r} = \frac{C_{\text{local}}}{\delta r} &\approx \frac{v}{T(f + f^{-1})} \left[ 1 - 2r \frac{\delta \log A}{\delta r} \right] \\ &\approx \frac{2v(2v - u)FG}{3T(F^2 + G^2)} \left[ \frac{4v^2 - 4v - u^2 + 28uv/3}{(1 - u)(2v - u)^2} - \frac{2}{1 - 2u - 2v/3} \right. \\ &\quad \left. + \frac{2 - 16v^3 + 4(7 - 13u)v^2 + 2(u - 1)(5u - 3)v - 9u(u - 1)(2u - 1)}{F^2 + G^2} \right]. \end{aligned} \quad (60)$$

Interestingly, the heat capacity is regular over all physically allowed values of  $(u, v) \neq \mathcal{R}$ . Even though they take both signatures, their values change smoothly.

##### B. $r_- \ll r_H$ approximation

Let us next consider the ‘almost sphere’ case which excises only the central singularity by using the limit  $r_- \rightarrow 0$ . We are interested in the solution other than the regular one,  $\nu \neq \nu_r$ . Solving Eqs. (12) and (13) around the center

(or simply quoting results in Ref. [26]), one gets approximately

$$m(r_-) = -\frac{\mu_0 r_H}{2} + \frac{\kappa r_H}{10} \left(\frac{r_-}{r_H}\right)^5, \quad u = -\frac{r_H \mu_0}{r_-}, \quad v = \frac{\kappa}{2} \left(\frac{r_-}{r_H}\right)^4 = \frac{\kappa \mu_0^4}{2} u^{-4}. \quad (61)$$

Note that there is a central conical singularity with negative mass at  $r = 0$  unless it is excised. By using this result, we get

$$A_- = \frac{v_-^{3/4} \sqrt{1-u_-}}{2v_- - u_-} \frac{F_-}{F_-^2 + G_-^2} = 4 \frac{v_-^{7/4}}{|u_-|^{7/2}} = 4 \left(\frac{v_-}{u_-^2}\right)^{7/4} = 4 \left(\frac{\kappa \mu_0^4}{2u_-^6}\right)^{7/4}.$$

Therefore,  $A_-(r_+/r_-)^{1/2} \ll 1$  in the limit. Now, the LHC and the heat capacity takes the form in Eqs. (45) and (54), respectively. Their explicit behaviors are shown on the right panel of Figs. 1 and 3, respectively.

Note that the form of the heat capacities are completely different from that of the regular solution. As for a regular solution,  $C_{\text{local}}^{\text{reg}}$  is singular on the line  $P$  and vanishes on  $H$ . On the other hand,  $C_{\text{local}}^P$  vanishes on the line  $P$  and is singular on  $H$  for the present case. This implies that the excision of the central conical singularity plays an important role in the thermodynamics of the system.

Let us observe the case that both boundaries are located around the center,  $r_- \ll r_+ \ll r_H$ . The entropy of the system is given by

$$S_{\text{rad}} = S_+ - S_- = \frac{r_H^{3/2}}{6} \left(\frac{8\pi\sigma}{\kappa\mu_0^2}\right)^{1/4} \frac{r_+ - r_-}{r_H} + \dots. \quad (62)$$

where

$$S_{\pm} = \frac{r_H^{3/2}}{3} \frac{e^{3\xi_{\pm}/2}}{\sqrt{1-u_{\pm}}} \left(\frac{4\pi\sigma}{v_{\pm}}\right)^{1/4} \left(\frac{2v_{\pm}}{3} + u_{\pm}\right) \approx \frac{r_H^{3/2}}{3} \left(\frac{8\pi\sigma\mu_0^2}{\kappa}\right)^{1/4} \left(-1 + \frac{1}{2\mu_0} \frac{r_{\pm}}{r_H} + \dots\right).$$

Note that  $S_{\pm}$  has a non-vanishing negative constant contribution in the  $r \rightarrow 0$  limit. The heat capacity of the system is independent of the information at the inner boundary and takes negative value,

$$C_{\text{local}} \approx C_V \approx -\frac{2\kappa^2 r_H}{\mu_0} \left(\frac{r_+}{r_H}\right)^{10} + \dots.$$

Therefore, the system must be unstable under perturbations. The heat capacity becomes positive after the solution curve passes the line  $P$ , where approximation  $r_+ \ll r_H$  does not hold any more.

### C. Near approximate horizon case

We next consider the case that  $r_-$  is located around the approximate horizon. We assume that the approximate horizon is about to form an event horizon,  $\nu \sim 0$ . A special case is that the inner boundary is located at the approximate horizon. Then, the (local) heat capacity is given by that of the regular solution as discussed in the paragraph just after Eq. (44).

First, let us consider both boundaries are located around the approximate horizon,  $1 - u_{\pm} \ll 1$  and  $\varepsilon^2 \ll v_+ < v_- \ll \varepsilon^{-2/3}$ , where  $\varepsilon = 9\nu/16$  is a small expansion parameter. In this region,<sup>1</sup> the solution curve  $C_{\nu}$  satisfies [26]

$$1 - u \approx \varepsilon \frac{(2v/3 + 1)^2}{\sqrt{2v}} + O(\varepsilon^2). \quad (63)$$

The radius is given by

$$r = r_H e^{\xi}; \quad \xi = \frac{\varepsilon}{\sqrt{2v}} \left(1 - \frac{v}{6}\right) - \frac{11\varepsilon}{12}, \quad (64)$$

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<sup>1</sup> This region corresponds to  $[\mathfrak{A}, \mathfrak{G}]$  in Ref. [26].

where we choose  $\xi = 0$  at  $H$ . Note that  $r$  changes only a bit for a large change of  $v$  in this region. This gives, by using  $r_+ \simeq r_- \simeq r_H$  and  $u \simeq 1$ ,

$$\frac{(2v-u)A}{\sqrt{r}} \approx -2^{-5/4} \sqrt{\frac{\varepsilon}{r_H v}}, \Rightarrow \mathfrak{A} = \sqrt{\frac{v_-}{v_+} \frac{2v_- - 1}{2v_+ - 1}}.$$

Therefore, the heat capacity and the LHC become

$$\chi^{-1} C_V \approx C_{\text{local}} = -\frac{\varepsilon r_+}{\sqrt{2} v_+ T_+} \left( \frac{2v_+}{3} + 1 \right) (\sqrt{v_+} - \sqrt{v_-}) [2(v_+ + v_- + \sqrt{v_+ v_-}) - 1]. \quad (65)$$

Because  $v_+ < v_-$ , the sign of the heat capacity is determined by the sign of  $2(v_+ + v_- + \sqrt{v_+ v_-}) - 1$ . For  $v_- \geq 1/2$ , the LHC is positive definite. The LHC becomes negative only if  $v_- < 1/2$  and  $v_+ < (\sqrt{2 - 3v_-} - \sqrt{v_-})/2$ .

We next consider the case that the both boundaries are located in the region outside the horizon satisfying  $v_{\pm} \ll 1$  and  $\varepsilon^{2/3} \ll u_+ < u_- < 1$ .<sup>2</sup> In this case,  $r$ ,  $u$  and  $v$  are related by

$$v = \frac{\varepsilon^2}{2u^2(1-u)^2}, \quad r \approx \frac{r_H}{u} \left( 1 - \frac{11\varepsilon}{12} \right) + \dots \quad (66)$$

Then, the function  $A$  is given by

$$\frac{A}{\sqrt{r}} \approx \frac{\varepsilon^{7/2}}{2^{3/4} r_H^{1/2}} \frac{2u-1}{u^6 \chi^{10}}.$$

Putting this to Eq. (44), the LHC becomes

$$C_{\text{local}} = \frac{\varepsilon^4 r_H}{T_+} \left( \frac{2u_+ - 1}{u_+^6 \chi_+^{10}} - \frac{2u_- - 1}{u_-^6 \chi_-^{10}} \right). \quad (67)$$

Because  $(2u-1)/(\chi^{10} u^6)$  is a monotonically increasing function of  $u$  and  $u_+ < u_-$ , the heat capacity is negative definite. The heat capacity becomes

$$C_V = \frac{r_H \chi_+^2}{\beta^{-1} u_+} \frac{u_-^6 \chi_-^{10}}{2u_- - 1} \left( \frac{2u_+ - 1}{u_+^6 \chi_+^{10}} - \frac{2u_- - 1}{u_-^6 \chi_-^{10}} \right). \quad (68)$$

The signature of the heat capacity is different from that of the local one but is dependent on the position of the inner boundary. Note also that the heat capacity is  $O(1)$  even though the local one is  $O(\varepsilon^4)$ . The heat capacity is never singular because  $u_- < 1/2$  always in this region.

Finally, we consider the case that the inner and the outer boundaries are located around the approximate horizon and outside of the approximate horizon, respectively. The LHC becomes

$$C_{\text{local}} = \frac{\varepsilon r_H}{T_+} \frac{1 - u_+}{-\varepsilon(1 - 2u_+) + \sqrt{2v_-}(2v_- - u_-)} \approx \frac{\varepsilon r_H}{T_+} \frac{1 - u_+}{\sqrt{2v_-}(2v_- - u_-)}. \quad (69)$$

In the first equality, we keep the first term in the denominator because  $C_{\text{local}}$  should be consistent with Eq. (43) when the inner boundary is on  $H$  as discussed just below Eq. (44). The value of LHC is  $O(\varepsilon)$ . However, it is  $O(1)$  if the inner boundary is on  $H$ . Therefore, the LHC is very sensitive on the change of the inner boundary around  $H$ . The heat capacity is given by

$$C_V = \frac{\varepsilon r_H}{\beta^{-1} \varepsilon u_+ + \sqrt{2v_-}(2v_- - u_-)} \frac{1}{\sqrt{2v_-}(2v_- - u_-)} \approx \frac{\varepsilon r_H}{\beta^{-1} \sqrt{2v_-}(2v_- - u_-)} \frac{1}{\sqrt{2v_-}(2v_- - u_-)}, \quad (70)$$

where the last equality is valid only when  $2v_- \neq u_-$ . Usually, the value of heat capacity is of  $O(\varepsilon)$ . If the inner boundary is located on  $H$ , the heat capacity suddenly jumps to  $O(1)$ , which value is the same as that of the regular solution in Eq. (51) as expected just after Eq. (53).

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<sup>2</sup> This corresponds to the region  $[\mathfrak{S}, \mathfrak{S}']$  in Ref. [26].

## V. SUMMARY AND DISCUSSIONS

In this work, we have studied analytically the heat capacity of a static spherically symmetric self-gravitating radiations in the context of general relativity. To avoid the ambiguity due to the central conical singularity, we excise the central region and introduce an inner boundary at  $r_-$ . Then, we assumed that the system inside the inner boundary is filled with matters of spherically symmetric distribution with its total mass  $M_-$  being held. Therefore, the radiations are confined inside a shell bounded by two stiff boundaries at  $r_+$  and  $r_-$ . The distribution of the radiations and the geometry are described by putting the boundary values on a solution curve  $C_\nu$  on a two dimensional plane ( $u = 2m(r)/r, v = 4\pi r^2 \rho$ ) of scale invariant variables, which curve can be found by solving the Tolman-Oppenheimer-Volkoff (TOV) equation.

We first derived how to relate the variations at the outer boundary with those at the inner boundary. At the outer boundary, there are three independent variables,  $r_+$ ,  $M_+$ , and  $\rho_+$ . To obtain the heat capacity of the radiations for fixed inner and outer boundaries, we have assumed that the location of the inner and the outer boundaries are held. Then at the outer boundary, there remains two independent variables which can be varied. Because  $M_-$  and  $r_-$  are held, only  $v_-$  can be varied at the inner boundary. The variation  $\delta v_-$  will induce the variations  $\delta u_+$  and  $\delta v_+$  at the outer boundary through the TOV equation. Because there are only one variation at the inner boundary, the variations at the outer boundary should be related and the relation shows up as a heat capacity. To get an analytic form for the heat capacity, we additionally use the thermodynamic identity  $\delta S_{\text{rad}} = \beta \delta M_{\text{rad}}$ , which is derived from the variation of the entropy formulae.

Let us display a few interesting results. First, we should emphasize that the local heat capacity (LHC) with respect to the local temperature at the outer boundary,  $T_+$ , is different from the heat capacity with respect to the globally measured temperature,  $\beta^{-1}$ . They can differ not only in their relative size, but also in their signatures. This is because the local temperature is dependent not only on  $\beta^{-1}$  but also on the ratio  $M_+/r_+$ . Second, we present a few interesting examples. There are two limiting forms for the (local) heat capacities. i) When the inner boundary is located at the approximate horizon, the heat capacity and the LHC of the shell of the radiations are the same as those of the self-gravitating sphere of regular solution. ii) When the inner boundary is located on the line  $P : 2u_- + 2v_-/3 = 1$ , the heat capacities show other limiting forms much different from that of the regular ones. The LHC for the spherical shell vanishes and diverges at the lines where that for the regular sphere diverges and vanishes, respectively. The heat capacity also show two distinct limiting forms in the same limits.

As the outer boundary changes, a general LHC is singular or null on curves on the  $(u_+, v_+)$  plane. The singular curve  $S$  gradually changes from the line  $P$  to  $H : u_+ = 2v_+$  as the condition on the inner boundary changes. On the other hand, the null curve  $N$  changes from  $H$  to  $P$ . When viewed from the clockwise direction centered on  $\mathcal{R}$ , the LHC takes positive definite value from  $N$  to  $S$ . Similarly, a general heat capacity is also singular or null on curves on the  $(u_+, v_+)$  plane. The singular curve  $S'$  changes from  $Q : 8v_+/3 + u_+ = 1$  to  $S^P : (1 - u_+)^2(2v_+ - u_+) + 4v_+^2(1 - 2u_+ - 2v_+/3) = 0$ . On the other hand, the null curve  $N'$  changes from  $H$  to  $P$  similarly to the case of the LHC. When viewed from the clockwise direction centered on  $\mathcal{R}$ , the heat capacity is positive definite from  $N'$  to  $S'$ .

For the case of the zero size limit of the inner boundary,  $r_- \rightarrow 0$ , it was shown that the (local) heat capacity does not go to the form of the regular solution. Rather, they approaches the opposite limit ii) unless the solution curve is that of that of regular one. Finally, we have obtained the heat capacity for the case that both boundaries are located around the approximate horizon. We find that there are no singularity of (local) heat capacity contrary to the general case.

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